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A boundary element method with analytical integration for deformation of inhomogeneous elastic materials

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Abstract. This paper is concerned with obtaining solutions to the equation governing static deformations of inhomogeneous elastic materials. The material parameters are assumed to vary continuously with the spatial variables. A boundary element method (BEM) with analytical integration is used to find the solutions. The results show that the BEM is feasible to be used to find the solutions of the problems and with analytical integration the BEM gives very accurate solutions in a shorter computation time.

1. Introduction

A various classes of problems have been solved numerically using BEM. The classes of problems include heat conduction problems, infiltration problems, deformation problems of elastic materials, pollutant transport problems and many others. The use of BEM has also been applied to homogeneous as well as to inhomogeneous materials, and not only to isotropic but also to anisotropic materials. For example, [1] used the BEM for solving infiltration problems of isotropic homogeneous media, [2] derived a fundamental solution, [3] considered diffusion-convection problems for anisotropic homogeneous materials, [4] solved elasticity problems for isotropic inhomogeneous materials and [5] studied elasticity problems, [6] worked on elliptic boundary value problems, [7] solved heat conduction problems, [8] worked on elasticity problems and [9] considered transient heat conduction problems for anisotropic inhomogeneous materials.

Apparently in the implementation of the BEM the partial differential equation which governs the system has to be converted to a boundary integral equation. And eventually after a discretisation of the boundary into a number of segments, integration over each segment has to be calculated either numerically or analytically if possible. To some extent analytical integration has of course advantages over numerical integration in the aspect of accuracy and computation time.

In recent years some progress has been made toward solving numerically the deformation problem of inhomogeneous elastic materials using the BEM. The work done by [4], for example, deals with the case of isotropic materials, whereas in [5, 8] considered the case for anisotropic materials. However, in all of the papers the authors used numerical integration for finding the BEM solutions. The current study is concerned with finding the solutions of the problems using the BEM with analytical integration.



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2. The boundary value problem ³

Referred to a Cartesian frame $Ox_1x_2x_3$ the equilibrium equations governing small deformations of an inhomogeneous anisotropic elastic material occupying a region ³ in R^3 with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed curves may be written in the form

$$\frac{\partial}{\partial x_j} \left[c_{ijkl}(\mathbf{x}) \frac{\partial u_k}{\partial x_l} \right] = 0 \quad (1)$$

where $i, j, k, l = 1, 2, 3$, $\mathbf{x} = (x_1, x_2, x_3)$, u_k denotes the displacement, $c_{ijkl}(\mathbf{x})$ the elastic parameters and the repeated summation convention (summing from 1 to 3) is used for repeated Latin suffices. Throughout the paper it is assumed that u_k is independent of x_3 and takes the form

$$u_k = A_k f(x_1 + \tau x_2) \quad (3)$$

where A_k , τ are constants and f is a twice differentiable function. The stress displacement relations are given by

$$\sigma_{ij} = c_{ijkl} \frac{\partial u_k}{\partial x_l}$$

and the traction vector P_i on the boundary $\partial\Omega$ is defined as

$$P_i \equiv \sigma_{ij} n_j = c_{ijkl} \frac{\partial u_k}{\partial x_l} n_j \quad (2)$$

where $\mathbf{n} = (n_1, n_2)$ denotes the outward pointing normal to the boundary $\partial\Omega$.

For all points in Ω the coefficients $c_{ijkl}(\mathbf{x})$ are required to satisfy the usual symmetry conditions and also sufficient conditions for the strain energy density to be positive. This requirement ensures that the system of partial differential equations is elliptic throughout Ω .

A solution to (1) is sought which is valid in the region Ω and satisfies the boundary conditions that on $\partial\Omega_1$ the displacement u_k is specified and on $\partial\Omega_2$ the traction P_i is specified, where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$.

³ Reduction to a linear constant coefficients equation

The coefficients in (1) are now required to take the form

$$c_{ijkl}(\mathbf{x}) = c_{ijkl}^{(0)} g(\mathbf{x}) \quad (3)$$

where the $c_{ijkl}^{(0)}$ are constants. Equation (1) thus may be written

$$c_{ijkl}^{(0)} \frac{\partial}{\partial x_j} \left[g(\mathbf{x}) \frac{\partial u_k}{\partial x_l} \right] = 0 \quad (4)$$

Consider the transformation

$$u_k = g^{-1/2} \psi_k \quad (5)$$

Use of (5) in (4) provides the equation

$$c_{ijkl}^{(0)} g^{1/2} \frac{\partial^2 \psi_k}{\partial x_j \partial x_l} - c_{ijkl}^{(0)} \psi_k \frac{\partial^2 g^{1/2}}{\partial x_j \partial x_l} = 0 \quad (6)$$

Thus if

$$c_{ijkl}^{(0)} \frac{\partial^2 \psi_k}{\partial x_j \partial x_l} = 0 \quad (7)$$

$$c_{ijkl}^{(0)} \frac{\partial^2 g^{1/2}}{\partial x_j \partial x_l} = 0 \quad (8)$$

3

then (6) will be satisfied. Thus when g satisfies the system (8) the transformation given by (5) transforms the linear system with variable coefficients (4) to the linear system with constant coefficients (7).

Equation (7) comprises a system of three constant coefficients linear elliptic partial differential equations in the three dependent variables u_k for $k = 1, 2, 3$. A solution to this system may be readily obtained in terms of arbitrary analytic functions.

As a result of the symmetry property $c_{ijkl} = c_{klij}$ equation (8) consists of a system of six constant coefficients partial differential equations in the one dependent variable $g^{1/2}$. In general the solution to this system consists of a linear function of the three independent variables x_1, x_2, x_3 . Thus in the general case g may be written in the form

$$g(\mathbf{x}) = (\alpha_i x_i)^2 \tag{9}$$

where the α_i are constants which may be used to fit the elastic parameters $c_{ijkl} = c_{ijkl}^{(0)} g(\mathbf{x})$ to given numerical data.

Although in the general case it is necessary for g to be restricted to the form given by (9) it is appropriate to note at this stage that for particular classes of deformations the transformation from (4) to (7) may be achieved for a more general class of functions $g(\mathbf{x})$. Some of these cases are now considered.

Now substitution of (3) and (5) into (2) yields

$$P_i = -P_{ik}^{[g]} \psi_k + P_i^{[\psi]} g^{1/2} \tag{10}$$

where

$$P_{ik}^{[g]} = c_{ijkl}^{(0)} \frac{\partial g^{1/2}}{\partial x_l} n_j \tag{11}$$

$$P_i^{[\psi]} = c_{ijkl}^{(0)} \frac{\partial \psi_k}{\partial x_l} n_j \tag{12}$$

The boundary integral equation for the solution of (7) may be written in the form

$$\eta \psi_m(\mathbf{x}_0) = - \int_{\partial\Omega} [\Phi_{im} P_i^{[\psi]} - \Gamma_{im} \psi_i] ds \tag{13}$$

Use of (5) and (10) in (13) yields

$$\eta g^{1/2} u_m(\mathbf{x}_0) = - \int_{\partial\Omega} [(g^{-1/2} \Phi_{im}) P_i - (g^{1/2} \Gamma_{km} - P_{ki}^{[g]} \Phi_{im}) u_k] ds$$

This equation provides a boundary integral equation for determining u_m and P_m at all points of Ω .

Also Φ_{im} and Γ_{im} are given by

$$\Phi_{im} = \frac{1}{2\pi} \Re \left\{ \sum_{\alpha=1}^3 A_{i\alpha} N_{\alpha k} \log(z_\alpha - c_\alpha) \right\} d_{km} \quad \Gamma_{im} = \frac{1}{2\pi} \Re \left\{ \sum_{\alpha=1}^3 L_{ij\alpha} N_{\alpha k} (z_\alpha - c_\alpha)^{-1} \right\} n_j d_{km} \tag{14}$$

where \Re denotes the real part of a complex number, $z_\alpha = x_1 + \tau_\alpha x_2$, and $c_\alpha = a + \tau_\alpha b$, where τ_α are the three roots with positive imaginary part of the polynomial in τ

$$|c_{i1k1}^{(0)} + c_{i2k1}^{(0)} \tau + c_{i1k2}^{(0)} \tau + c_{i2k2}^{(0)} \tau^2| = 0$$

The $A_{i\alpha}$ occurring in (14) are the solutions of the system

$$\left(c_{i1k1}^{(0)} + c_{i2k1}^{(0)}\tau_\alpha + c_{i1k2}^{(0)}\tau_\alpha + c_{i2k2}^{(0)}\tau_\alpha^2\right) A_{k\alpha} = 0$$

Also the $N_{\alpha k}$, $L_{ij\alpha}$ and d_{km} are defined by

$$\delta_{ik} = \sum_{\alpha} A_{i\alpha} N_{\alpha k} \quad L_{ij\alpha} = (c_{ijkl}^{(0)} + \tau_\alpha c_{ijk2}^{(0)}) A_{k\alpha} \quad \delta_{im} = -\frac{1}{2} i \sum_{\alpha=1}^3 \left\{ L_{i2\alpha} N_{\alpha k} - \bar{L}_{i2\alpha} \bar{N}_{\alpha k} \right\} d_{km}$$

where bar denotes the complex conjugate.

4. A further perturbation method

In this section a procedure is obtained when the coefficients $c_{ijkl}(\mathbf{x})$ is perturbed about its expression presented in (3) while retaining the condition (8). Specifically the coefficient $c_{ijkl}(\mathbf{x})$ is supposed to have the form

$$c_{ijkl}(\mathbf{x}) = c_{ijkl}^{(0)} g(\mathbf{x}) + \epsilon c_{ijkl}^{(1)}(\mathbf{x}) \quad (15)$$

where the $c_{ijkl}^{(0)}$ are constants, ϵ is a small parameter, $c_{ijkl}^{(1)}$ is a differentiable function and $g(\mathbf{x})$ satisfies (8).

Substitution of (15) into (1) and use of the transformation (5) give

$$c_{ijkl}^{(0)} \frac{\partial}{\partial x_j} \left[g \frac{\partial}{\partial x_l} (g^{-1/2} \psi_k) \right] = -\epsilon \frac{\partial}{\partial x_j} \left[c_{ijkl}^{(1)} \frac{\partial}{\partial x_l} (g^{-1/2} \psi_k) \right] \quad (16)$$

Use the analysis used to derive (7) from (4) for the left hand side and a simplification for the right hand side of (16) yields

$$c_{ijkl}^{(0)} \frac{\partial^2 \psi_k}{\partial x_j \partial x_l} = -\epsilon \left[\frac{\partial A_{ijk}}{\partial x_j} \psi_k + (A_{ijk} + \frac{\partial B_{ijkl}}{\partial x_l}) \frac{\partial \psi_k}{\partial x_j} + B_{ijkl} \frac{\partial^2 \psi_k}{\partial x_j \partial x_l} \right] \quad (17)$$

where

$$A_{ijk}(\mathbf{x}) = c_{ijkl}^{(1)} \frac{\partial g^{-1/2}}{\partial x_l}, \quad B_{ijkl}(\mathbf{x}) = c_{ijkl}^{(1)} g^{-1/2}$$

A solution to equation (17) is sought in the form

$$\psi_k(\mathbf{x}) = \sum_{r=0}^{\infty} \epsilon^r \psi_k^{(r)}(\mathbf{x}) \quad (18)$$

Substitution of (18) into (17) and equating coefficients of powers of ϵ yields

$$c_{ijkl}^{(0)} \frac{\partial^2 \psi_k^{(0)}}{\partial x_j \partial x_l} = 0 \quad (19)$$

$$c_{ijkl}^{(0)} \frac{\partial^2 \psi_k^{(r)}}{\partial x_j \partial x_l} = - \left[\frac{\partial A_{ijk}}{\partial x_j} \psi_k^{(r-1)} + (A_{ijk} + \frac{\partial B_{ijkl}}{\partial x_l}) \frac{\partial \psi_k^{(r-1)}}{\partial x_j} + B_{ijkl} \frac{\partial^2 \psi_k^{(r-1)}}{\partial x_j \partial x_l} \right] \quad (20)$$

for $r = 1, 2, \dots$

The integral equations for (19) and (20) are respectively

$$\eta\psi_m^{(0)}(\mathbf{x}_0) = - \int_{\partial\Omega} \left[\Phi_{im}P_i^{[\psi^{(0)}]} - \Gamma_{im}\psi_i^{(0)} \right] ds \tag{21}$$

where $P_i^{[\psi^{(0)}]} = c_{ijkl}^{(0)}(\partial\psi_k^{(0)}/\partial x_l)n_j$ and

$$\eta\psi_m^{(r)}(\mathbf{x}_0) = - \int_{\partial\Omega} \left[\Phi_{im}P_i^{[\psi^{(r)}]} - \Gamma_{im}\psi_i^{(r)} \right] ds + \int_{\Omega} h_i^{(r)}\Phi_{im}dS \tag{22}$$

where $P_i^{[\psi^{(r)}]} = c_{ijkl}^{(0)}(\partial\psi_k^{(r)}/\partial x_l)n_j$ and $h_i^{(r)}$ is the right hand side of (20) for $r = 1, 2, \dots$

The corresponding value of P may be written as

$$P_i = -P_{ik}^{[g]}\psi_k^{(0)} + P_i^{[\psi^{(0)}]}g^{1/2} + \sum_{r=1}^{\infty} \epsilon^r \left[-P_{ik}^{[g]}\psi_k^{(r)} + P_i^{[\psi^{(r)}]}g^{1/2} + G_i^{(r)} \right]$$

where $P_{ik}^{[g]}$ and $P_i^{[\psi]}$ are given by (11) and (12), and

$$G_i^{(r)} = c_{ijkl}^{(1)} \frac{\partial(g^{-1/2}\psi_k^{(r-1)})}{\partial x_l} n_j \tag{23}$$

To satisfy the boundary conditions in Section 2 it is required that $\psi_k^{(0)} = g^{1/2}u_k$ where u_k takes on its specified value on $\partial\Omega_1$. Also it is required that on $\partial\Omega_2$ $P_i^{[\psi^{(0)}]} = g^{-1/2}(P_i + P_{ik}^{[g]}\psi_k^{(0)})$ where P_i takes on its specified value on $\partial\Omega_2$. It then follows from (12) and (18) that $\psi_i^{(r)} = 0$ on $\partial\Omega_1$ and $P_i^{[\psi^{(r)}]} = g^{-1/2}(P_{ik}^{[g]}\psi_k^{(r)} - G_i^{(r)})$ on $\partial\Omega_2$ for $r = 1, 2, \dots$

The integral equations (21) and (22) may thus be written in the form

$$\begin{aligned} \eta\psi_m^{(0)}(\mathbf{x}_0) &= - \int_{\partial\Omega_1} \left[\Phi_{im}P_i^{[\psi^{(0)}]} - (\Gamma_{im}g^{1/2})u_i \right] ds \\ &\quad - \int_{\partial\Omega_2} \left[(g^{-1/2}\Phi_{im})P_i - (\Gamma_{km} - g^{-1/2}P_{ki}^{[g]}\Phi_{im})\psi_k^{(0)} \right] ds \end{aligned} \tag{24}$$

$$\begin{aligned} \eta\psi_m^{(r)}(\mathbf{x}_0) &= - \int_{\partial\Omega_1} \Phi_{im}P_i^{[\psi^{(r)}]} ds \\ &\quad - \int_{\partial\Omega_2} \left[-(g^{-1/2}\Phi_{im})G_i^{(r)} - (\Gamma_{km} - g^{-1/2}P_{ki}^{[g]}\Phi_{im})\psi_k^{(r)} \right] ds \\ &\quad + \int_{\Omega} h_i^{(r)}\Phi_{im}dS \quad \text{for } r = 1, 2, \dots \end{aligned} \tag{25}$$

The boundary integral equation (24) serves to determine the unknown $\psi_i^{(0)}(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega_2$ and the unknown $P_i^{[\psi^{(0)}]}(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega_1$. Once these boundary values are known equation (24) serves to determine $\psi_i^{(0)}$ and its derivatives for all points in the domain Ω .

Similarly the integral equation (25) serves to determine the unknown $\psi_i^{(r)}(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega_2$ and the unknown $P_i^{[\psi^{(r)}]}(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega_1$ for $r = 1, 2, \dots$. Once these boundary values are known equation (25) serves to determine $\psi_i^{(r)}$ and its derivatives for all points in the domain Ω . At each stage in using (25) to determine $\psi_i^{(r)}$ and the $G_i^{(r)}$ occurring in (25) may be obtained from (23) which may be evaluated from the previously determined derivatives of $\psi_i^{(r-1)}$.

Having determined the values of $\psi_i^{(r)}$ for all r equations (18) and (5) then provide successively, the values of $\psi_i(\mathbf{x})$ and $u_i(\mathbf{x})$.

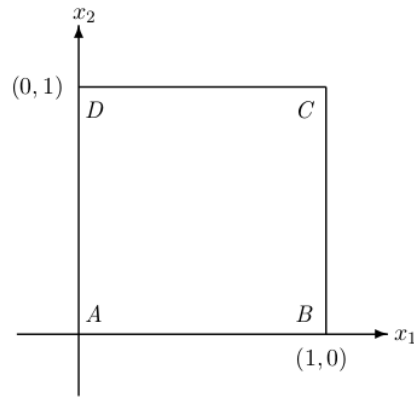


Figure 1. The geometry for the numerical examples

5. Numerical results

To show the usefulness of the formulas and the validity of the techniques derived in the previous sections some particular plane strain problems, for which the displacements $u_1 = u_1(x_1, x_2)$, $u_2 = u_2(x_1, x_2)$ and $u_3 = 0$, of inhomogeneous anisotropic materials will be considered.

The constant BEM is used to find numerical solutions to the problems. The domain Ω is taken to be a unit square (see Figure 1). Each side of the boundary $\partial\Omega$ is divided into N segments of equal length and nodal points are taken to be the mid points of each segment.

To save the computation time and to increase the accuracy, the evaluation of the line integrals of Φ_{im} , Γ_{im} and its derivatives along a segment $[\mathbf{q}_{k-1}, \mathbf{q}_k]$ joining the point \mathbf{q}_{k-1} to the point \mathbf{q}_k and excluding the singular points of Φ_{im} and Γ_{im} is done analytically by utilising the following results,

$$\int_{\mathbf{q}_{k-1}}^{\mathbf{q}_k} \log(z_\alpha - c_\alpha) ds(\mathbf{x}) = R_k \left[-1 + \log(A_\alpha + B_\alpha) + \frac{A_\alpha}{B_\alpha} \log\left(1 + \frac{B_\alpha}{A_\alpha}\right) \right],$$

$$\int_{\mathbf{q}_{k-1}}^{\mathbf{q}_k} (z_\alpha - c_\alpha)^{-1} ds(\mathbf{x}) = \frac{R_k}{B_\alpha} \log\left(1 + \frac{B_\alpha}{A_\alpha}\right)$$

$$\int_{\mathbf{q}_{k-1}}^{\mathbf{q}_k} \frac{\partial}{\partial a} [\log(z_\alpha - c_\alpha)] ds(\mathbf{x}) = \frac{-R_k}{B_\alpha} \log\left(1 + \frac{B_\alpha}{A_\alpha}\right)$$

$$\int_{\mathbf{q}_{k-1}}^{\mathbf{q}_k} \frac{\partial}{\partial b} [\log(z_\alpha - c_\alpha)] ds(\mathbf{x}) = \frac{-R_k \tau_\alpha}{B_\alpha} \log\left(1 + \frac{B_\alpha}{A_\alpha}\right)$$

$$\int_{\mathbf{q}_{k-1}}^{\mathbf{q}_k} \frac{\partial}{\partial a} [(z_\alpha - c_\alpha)^{-1}] ds(\mathbf{x}) = R_k [A_\alpha (A_\alpha + B_\alpha)]^{-1}$$

$$\int_{\mathbf{q}_{k-1}}^{\mathbf{q}_k} \frac{\partial}{\partial b} [(z_\alpha - c_\alpha)^{-1}] ds(\mathbf{x}) = R_k \tau_\alpha [A_\alpha (A_\alpha + B_\alpha)]^{-1}$$

where

$$\mathbf{q}_k = (x_k, y_k), \quad x = x_1, \quad y = x_2 \quad R_k = |\mathbf{q}_k - \mathbf{q}_{k-1}|$$

$$A_\alpha = x_{k-1} - a + \tau_\alpha (y_{k-1} - b) \quad B_\alpha = x_k - x_{k-1} + \tau_\alpha (y_k - y_{k-1})$$

The last four formulas are used for the calculation of the stress distribution in the domain Ω . Along a segment $[\mathbf{q}_{k-1}, \mathbf{q}_k]$ which includes a singular point (ie. the nodal or mid point of a

segment) the evaluation of the integral of Φ_{im} is done by splitting the segment into two parts on either side of the singular point and summing the values of integrals along both sub segments. This then gives the following formula

$$\int_{\mathbf{q}_{k-1}}^{\mathbf{q}_k} \log(z_\alpha - c_\alpha) ds(\mathbf{x}) = R_k(-1 + \log \frac{1}{2} + \log B_\alpha)$$

Meanwhile the integral of Γ_{im} is zero due to the fact that in this case the vectors $\mathbf{x} - \mathbf{x}_0$ and \mathbf{n} are perpendicular and therefore the normal derivative of Φ_{im} (and so Γ_{im}) is zero.

For the methods requiring domain integrals the domain Ω is discretised into N^2 equal sub-squares and the following analytical results are used for the evaluation of area integrals over a particular sub square $\{(x_1, x_2) : s \leq x_1 \leq t, v \leq x_2 \leq w\}$

$$\begin{aligned} \int_s^t \int_v^w \log [x_1 - a + \tau_\alpha(x_2 - b)] dx_2 dx_1 = & \\ q_1 \left[-(v-b)(t-a) - \frac{1}{2\tau_\alpha}(t-a)^2 - \frac{1}{2}\tau_\alpha(v-b)^2 \right] + & \\ q_2 \left[(v-b)(s-a) + \frac{1}{2\tau_\alpha}(s-a)^2 + \frac{1}{2}\tau_\alpha(v-b)^2 \right] + & \\ q_3 \left[-(w-b)(s-a) - \frac{1}{2\tau_\alpha}(s-a)^2 - \frac{1}{2}\tau_\alpha(w-b)^2 \right] + & \\ q_4 \left[(w-b)(t-a) + \frac{1}{2\tau_\alpha}(t-a)^2 + \frac{1}{2}\tau_\alpha(w-b)^2 \right] - & \\ \frac{3}{2}(t-s)(w-v) & \end{aligned}$$

$$\begin{aligned} \int_s^t \int_v^w x_1 \log [x_1 - a + \tau_\alpha(x_2 - b)] dx_2 dx_1 = & \\ \frac{1}{12\tau_\alpha} \left[q_1 \{ 2q_5^3 - 6aq_5^2 + 6(a^2 - t^2)q_5 - 2a^3 - 4t^3 + 6at^2 \} + & \\ q_2 \{ -2q_5^3 + 6aq_5^2 - 6(a^2 - s^2)q_5 + 2a^3 + 4s^3 - 6as^2 \} + & \\ q_3 \{ 2q_6^3 - 6aq_6^2 + 6(a^2 - s^2)q_6 - 2a^3 - 4s^3 + 6as^2 \} + & \\ q_4 \{ -2q_6^3 + 6aq_6^2 - 6(a^2 - t^2)q_6 + 2a^3 + 4t^3 - 6at^2 \} + & \\ \tau_\alpha q_7 \{ -4b + 2(v+w) \} + q_7 \{ -4a - 7(s+t) \} \right] & \end{aligned}$$

$$\begin{aligned} \int_s^t \int_v^w x_2 \log [x_1 - a + \tau_\alpha(x_2 - b)] dx_2 dx_1 = & \\ \frac{-1}{12\tau_\alpha^2} \left[q_1 \{ 2\tau_\alpha^3 q_8 + 6\tau_\alpha^2(v^2 - b^2)(t-a) + 6\tau_\alpha b(t-a)^2 - 2(t-a)^3 \} + & \\ q_2 \{ -2\tau_\alpha^3 q_8 - 6\tau_\alpha^2(v^2 - b^2)(s-a) - 6\tau_\alpha b(s-a)^2 + 2(s-a)^3 \} + & \\ q_3 \{ 2\tau_\alpha^3 q_9 + 6\tau_\alpha^2(w^2 - b^2)(s-a) + 6\tau_\alpha b(s-a)^2 - 2(s-a)^3 \} + & \\ q_4 \{ -2\tau_\alpha^3 q_9 - 6\tau_\alpha^2(w^2 - b^2)(t-a) - 6\tau_\alpha b(t-a)^2 + 2(t-a)^3 \} + & \\ \tau_\alpha q_7 \{ 4b + 7(v+w) \} + q_7 \{ 4a - 2(s+t) \} \right] & \end{aligned}$$

where

$$\begin{aligned} q_1 &= \log [t - a + \tau_\alpha(v - b)] & q_2 &= \log [s - a + \tau_\alpha(v - b)] & q_3 &= \log [s - a + \tau_\alpha(w - b)] \\ q_4 &= \log [t - a + \tau_\alpha(w - b)] & q_5 &= \tau_\alpha(v - b) & q_6 &= \tau_\alpha(w - b) \\ q_7 &= \tau_\alpha(t - s)(w - v) & q_8 &= b^3 + 2v^3 - 3v^2b & q_9 &= b^3 + 2w^3 - 3w^2b \end{aligned}$$

Table 1. Numerical and analytical results

Position (x_1, x_2)	BEM 12 segments		BEM 40 segments		Analytical	
	u_1/\bar{u}	u_2/\bar{u}	u_1/\bar{u}	u_2/\bar{u}	u_1/\bar{u}	u_2/\bar{u}
(0.1,0.5)	0.0542	0.1477	0.0493	0.1486	0.0495	0.1485
(0.3,0.5)	0.1469	0.2450	0.1456	0.2428	0.1456	0.2427
(0.5,0.5)	0.2383	0.3333	0.2381	0.3333	0.2381	0.3333
(0.7,0.5)	0.3261	0.4181	0.3271	0.4204	0.3271	0.4205
(0.9,0.5)	0.4086	0.5047	0.4130	0.5043	0.4128	0.5046

5.1. A test problem for Section 4

Consider the isotropic case for which the elastic moduli take the form (15) with $A/\bar{c} = \lambda^{(0)} + 2\mu^{(0)}$, $B/\bar{c} = \lambda^{(0)}$, $D/\bar{c} = \lambda^{(0)} + 2\mu^{(0)}$ and $F/\bar{c} = \mu^{(0)}$,

$$\begin{aligned} \epsilon = 0.1 \quad g(\mathbf{x}) &= (1 + 0.1x_1/l)^2 & \lambda^{(0)} = \mu^{(0)} &= 0.5 \\ \lambda^{(1)} &= (1 + 0.1x_1/l)^2 & \mu^{(1)} &= 0 \end{aligned}$$

and the boundary conditions

$$\begin{array}{lll} u_1/\bar{u} = 5 - 5/(1 + 0.1x_1/l) & u_2/\bar{u} = 5 - 4.9/(1 + 0.1x_1/l) & \text{on AB} \\ u_1/\bar{u} = 0.4545 & u_2/\bar{u} = 0.5454 & \text{on BC} \\ u_1/\bar{u} = 5 - 5/(1 + 0.1x_1/l) & u_2/\bar{u} = 5 - 4.9/(1 + 0.1x_1/l) & \text{on CD} \\ u_1/\bar{u} = 0 & u_2/\bar{u} = 0.1 & \text{on AD} \end{array}$$

This problem admits the analytical solution $u_1/\bar{u} = 5 - 5/(1 + 0.1x_1/l)$ and $u_2/\bar{u} = 5 - 4.9/(1 + 0.1x_1/l)$.

An approximate solution is taken to be the sum of the first two terms of the series solution (18). The so called regular method is used to avoid a singular point by locating the source points outside the domain at the position of a distance as long as the length of the boundary segment measured from the mid point of each boundary segment. It should be noted that $\eta = 0$ if $\mathbf{x}_0 \notin \Omega$. The area integral over a domain cell Ω_l is approximated with

$$\int_{\Omega_l} h_i^{(r)} \Phi_{im} dS \approx h_i^{(r)}(\mathbf{x}) \int_{\Omega_l} \Phi_{im} dS$$

where \mathbf{x} is the centre point of Ω_l .

Table 1 compares the analytical and BEM results for some interior points. It is observed that the accuracy improves as the number of segments increases. It also improves as we move further from the boundary points to the centre point of the domain.

6. Summary

Some BEMs for the solution of certain classes of boundary value problems of elasticity for anisotropic inhomogeneous media has been derived. The methods are generally easy to implement to obtain numerical values for particular problems. They can be applied to a wide class of important practical problems for inhomogeneous anisotropic materials. The numerical results obtained using the methods indicate that they can provide accurate numerical solutions.

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